STATE SPACE DECOMPOSITION FOR NONAUTONOMOUS DYNAMICAL SYSTEMS

XIAOPENG CHEN AND JINQIAO DUAN

ABSTRACT. Decomposition of state spaces into dynamically different components is helpful for the understanding of dynamical behaviors of complex systems. A Conley type decomposition theorem is proved for nonautonomous dynamical systems defined on a non-compact but separable state space. Namely, the state space can be decomposed into a chain recurrent part and a gradient-like part.

This result applies to both nonautonomous ordinary differential equations on Euclidean space (which is only locally compact), and nonautonomous partial differential equations on infinite dimensional function space (which is not even locally compact). This decomposition result is demonstrated by discussing a few concrete examples, such as the Lorenz system and the Navier-Stokes system, under time-dependent forcing.

1. Introduction

The decomposition of state spaces for dynamical systems or flows is desirable for better understanding of dynamical behaviors. The Conley decomposition theorem [8] says that any flow on a *compact* state space decomposes the space into a chain recurrent part and a gradient-like part. The theorem describes the dynamical behavior of each point in the systems. It is considered as a fundamental theorem of dynamical systems [23].

There are two essential concepts in the consideration of Conley decomposition of state spaces. One is the chain recurrence set. Conley [8] showed that the chain recurrent set $CR(\varphi)$ for a dynamical system φ on a compact state space can be represented in terms of complement sets of (local) attractors. This result is widely studied and further extended by others in different contexts [6, 7, 10, 14, 15, 16, 24] or for random dynamical systems [20, 21, 22].

The other essential concept is the so-called complete Lyapunov function, which quantifies gradient-like behavior. A complete Lyapunov function for a dynamical system φ is a continuous, real-valued function L defined on the state space which is strictly decreasing on orbits outside the chain redurrent set and such that: (a) The range $L(CR(\varphi))$ is

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nowhere dense; (b) If c belongs to the range $L(CR(\varphi))$, then $L^{-1}(c)$ is a component of the chain recurrent set. We call the complement of the chain recurrent set the gradient-like part of the flow. For more details see [8]. Furthermore, the complete Lyapunov function can be extended to non-compact state spaces for deterministic dynamical systems [15, 17, 24, 25] or random dynamical systems [21, 22]. Especially, in [21], the base space needs to be separable to construct the countable local attractors. While in [22], it is the weak complete Lyapunov function for the random semiflows.

In the present paper, we consider Conley type decomposition for nonautonomous dynamical systems (NDS), defined on not necessarily compact state spaces. Recall that a nonautonomous dynamical system is defined in terms of a cocycle mapping on a state space that is driven by an *autonomous* dynamical system acting on a base space. More details about nonautonomous dynamical systems are reviewed in the next section. The standard examples of nonautonomous dynamical systems are those generated by nonautonomous ordinary or partial differential equations, which arising from modeling in biological, physical and environmental systems. We will prove the following main result.

Theorem 1.1. *(Conley decomposition for NDS).*
A nonautonomous dynamical system with a separable (but not necessarily compact) state space decomposes the space into a chain recurrent part and a gradient-like part.

Here the "gradient-like part" for the NDS is indicated by the complete Lyapunov function. Such function is constructed using special attractor-repeller pairs. It is known that the attractor-repeller pairs are basic notions for the definition of Morse decompositions [9, 27, 28]. The Conley type state space decomposition for NDS can be applied to both nonautonomous ordinary differential equations on Euclidean space (which is only locally compact), and nonautonomous partial differential equations on infinite dimensional function space (which is not even locally compact). In Section 5, we illustrate this result by discussing a few concrete examples, such as the Lorenz system and the Navier-Stokes system, under time-dependent forcing.

To prove the above decomposition theorem, we first define and investigate the chain recurrent set for a nonautonomous dynamical system. In this context, a local attractor is a pullback attractor when it is a nonempty compact subset in the state space. The relationship of different attractors for nonautonomous dynamical systems is considered in [4, 18]. In the case of nonautonomous ordinary or partial differential equations, we can consider chain recurrent set for the corresponding skew-product dynamical system [3, 26]. It is known that the global attractor for a skew-product dynamical system corresponds to the pullback attractor on the state space [4, 5, 32]. We prove a similar relation for local attractors (Lemma 3.7) and apply to the chain recurrent set. Then, we consider the complete Lyapunov function for NDS. This concept of Lyapunov functions

is weaker than that for autonomous dynamical systems. Note that the base space here does not need to be separable.

This paper is organized as follows. After reviewing basic facts for nonautonomous dynamical systems (NDS) in Section 2, we investigate chain recurrent sets and complete Lyapunov functions for NDS in Section 3 and Section 4, respectively. The nonautonomous decomposition Theorem 1.1 is thus proved. In Section 5 we present a few examples, both ordinary and partial differential equations, to demonstrate the decomposition result.

2. Preliminineries

We will use the symbol $\mathbb{T}$ for either $\mathbb{R}$ or $\mathbb{Z}$, and denote by $\mathbb{T}^+$ all non-negative elements of $\mathbb{T}$. Let $dist_X$ denote the Hausdorff semi-metric between two nonempty sets of a metric space (X, d_X), that is

$$dist_X(A, B) = \sup_{a\in A} \inf_{b\in B} d_X(a, b), \tag{2.1}$$

for $A \subset X, B \subset X$. In addition, if A or B are empty, we set $dist_X(A, B) = 0$. We recall some basic definitions for nonautonomous dynamical systems [4, 27, 29] on state space X with base space (also a metric space) P.

Definition 2.1. *(Nonautonomous Dynamical System (NDS)). An autonomous dynamical system $(P, \mathbb{T}, \theta)$ on P consists of a continuous mapping $\theta_t : \mathbb{T} \times P \to P$ for which the $\theta_t = \theta(t, \cdot) : P \to P$, $t \in \mathbb{T}$, form a group of homeomorhpisms on P under composition over $\mathbb{T}$, that is, satisfy*

$$\theta_0 = id_P, \qquad \theta_{t+s} = \theta_t \cdot \theta_s$$

for all $t, s \in \mathbb{T}$. In addition, a continuous mapping $\varphi : \mathbb{T}^+ \times P \times X \to X$ is called a cocycle with respect to an autonomous dyanmical system $(P, \mathbb{T}, \theta)$ if it satisfies

$$\varphi(0, p, x) = x, \quad \varphi(t + s, p, x) = \varphi(t, \theta_s p, \varphi(t, p, x))$$

for all $t, s \in \mathbb{T}^+$ and $(p, x) \in P \times X$.

The triple $< X, \varphi, (P, \mathbb{T}, \theta) >$ is called a nonautonomous dynamical system [4, 29]. Let $(\mathbb{U}, d_{\mathbb{U}})$ be the cartesian product of (P, d_P) and (X, d_X). Then the mapping $\pi : \mathbb{T}^+ \times \mathbb{U} \to \mathbb{U}$ defined by

$$\pi(t, (p, x)) := (\theta_t p, \varphi(t, p, x))$$

forms a semi-group on $\mathbb{U}$ over $\mathbb{T}^+$; see [31].

A subset M of $\mathbb{U}$ is called a nonautonomous set. Let $M(p) := \{x \in X : (p, x) \in M\}$ for $p \in P$. A nonautonomous set M is called closed, compact or open if $M(p), p \in P$, are closed, compact or open, respectively. A nonautonomous set M is called forward invariant if $\varphi(t, p, M(p)) \subset M(\theta_t p)$ for all $t \in \mathbb{T}^+$, $p \in P$ and backward invariant if

$\varphi(t,p,M(p)) \supset M(\theta_t p)$ for all $t \in \mathbb{T}^+$, $p \in P$. A nonautonomous set M is called invariant if $\varphi(t,p,M(p)) = M(\theta_t p)$ for all $t \in \mathbb{T}^+$ and $p \in P$.

Let $\mathscr{D}$ be a family of sets $(D(p))_{p\in P}$ in X such that if $(D(p))_{p\in P} \in \mathcal{D} \subset \mathscr{D}$, $(D'(p))_{p\in P} \in \mathscr{D}$ and $D'(p) \subset D(p)$, then $(D'(p))_{p\in P} \in \mathcal{D}$.

Definition 2.2. *(Pullback attractor for NDS).*
Let $< X, \varphi, (P,T,\theta) >$ be a nonautonomous dynamical system. An element $A(p) \in \mathcal{D}$ such that $A(p)$ is compact is called pullback attractor (with respect to $\mathcal{D}$) if the invariance property

$$\varphi(t,p,A(p)) = A(\theta_t p), \qquad t \in \mathbb{T}^+, p \in P$$

and the pullback convergence property

$$\lim_{t\to\infty} dist_X(\varphi(t,\theta_{-t}p, D(\theta_{-t}p)), A(p)) = 0$$

for all $(D(p))_{p\in P} \in \mathcal{D}$ are fulfilled.

In the following we recall the definitions of local attractor and chain recurrent set, which are important to the state space decomposition.

Definition 2.3. *(Local attractor for NDS).*
An open set $U(p)$ is called a pre-attractor if it satisfies

$$\overline{\bigcup_{t\geq\tau(p)} \varphi(t,\theta_{-t}p)U(\theta_{-t}p)} \subset U(p) \qquad \text{for some } \tau(p) > 0, \tag{2.2}$$

where τ is a function. We define the local attractor $A(p)$ inside $U(p)$ as follows:

$$A(p) = \bigcap_{n\in\mathbb{N}} \overline{\bigcup_{s\geq n\tau(p)} \varphi(s,\theta_{-s}p)U(\theta_{-s}p)}. \tag{2.3}$$

The basin of attraction $B(A,U)(p)$, determined by $A(p)$ and $U(p)$, is defined as follows:

$$B(A,U)(p) = \{x : \varphi(t,p)x \in U(\theta_t p) \text{ for some } t \geq 0\}. \tag{2.4}$$

It can be proved that if the time $\mathbb{T}$ is two-sided and the state space X is compact, then for any $D(p) \subset B(A,U)(p)$, we have

$$\lim_{t\to\infty} dist_X(\varphi(t,\theta_{-t}p, D(\theta_{-t}p)), A(p)) = 0.$$

For a given local attractor $A(p)$, we define the repeller, corresponding to $A(p)$, as $R(p) := X - B(A,U)(p)$. We call the pair (A,R) an attractor-repeller pair. Observe that attractor-repeller pair depends on the pre-attractor $U(p)$. We allow $A(p) = \emptyset$ or $R(p) = \emptyset$. We use $F(P \times X)$ to denote the set of all maps from $P \times X$ to $\mathbb{R}^+$ and is continuous at fixed $p \in P$.

Definition 2.4. *(Chain recurrent set for NDS).*
(a) For a given $\varepsilon \in F(P \times X), T(p) > 0$, the sequence $\{x_1(p), \cdots, x_n(p), x_{n+1}(p); t_1, t_2, \cdots, t_n\}$ is called an $(\varepsilon, T)(p)$-chain for φ from $x(p)$ to $y(p)$ if for $1 \leq i \leq n$

$$x_1(p) = x(p), \quad x_{n+1}(p) = y(p), \quad t_i \geq T(p).$$

and

$$d_X(\varphi(t_i, \theta_{-t_i}p)x_i(\theta_{-t_i}p), x_{i+1}(p)) < \varepsilon(p, \varphi(t_i, \theta_{-t_i}p)x_i(\theta_{-t_i}p)),$$

where $x_i(p)$ is the map from P to X.
(b) A map x from P to X is called chain recurrent if there exists an $(\varepsilon, T)(p)$-chain beginning and ending at $x(p)$ for any $\varepsilon \in F(P \times X), T(p) > 0$.

(c) We denote $CR_\varphi(p)$ the chain recurrent set for φ, i.e.,

$$CR_\varphi(p) = \{x(p) \mid x(p) \text{ is chain recurrent variable } \}.$$

Example 2.5. *Recall that a mapping $\gamma^* : P \to \mathcal{G}$ is called a generalized fixed point of the cocycle Φ if*

$$\Phi(t, p, \gamma^*(p)) = \gamma^*(\theta_t p) \text{ for } t \in \mathbb{R}^+.$$

Such a generalized fixed point for the NDS Φ is chain recurrent with respect to P(for more details see [13]).

Definition 2.6. *(Stationary solution for NDS).*
A solution x(p) is called stationary for NDS φ if $\varphi(t, p, x(p)) = x(\theta_t p)$.

From the Definition 2.6 we know that the stationary solution for NDS is in the chain recurrent set. In fact, we know from the definition that a generalized fixed point for NDS is also a stationary solution for NDS. This is the same as the random case [30]. If $x(p)$ and $y(p)$ are chain recurrent with respect to P. We say $x(p) \sim y(p)$ if and only if for each $\varepsilon \in F(P \times X)$, $T(p) > 0$ there is an (ε, T)(p)-chain from $x(p)$ and $y(p)$ and one from $y(p)$ to $x(p)$ for $p \in P$. The equivalence classes are called the chain transitive components of φ.

Definition 2.7. *(Complete Lyapunov function for NDS)*
A complete Lyapunov function for a NDS φ is a function $L : P \times X \mapsto \mathbb{R}^+$, with $L(p, \cdot)$ being continuous for $p \in P$, that satisfies the following conditions:
(a) If $x \in CR_\varphi(p)$, then

$$L(\theta_t p, \varphi(t, p)x) = L(p, x), \qquad \forall t > 0;$$

(b) If $x \in X - CR_\varphi(p)$, then

$$L(\theta_t p, \varphi(t, p)x) \leq L(p, x), \qquad \forall t > 0;$$

(c) The range of $L(p, \cdot)$ on $CR_\varphi(p)$ is a compact nowhere dense subset of $[0, 1]$;
(d) If $x(p)$ and $y(p)$ belong to the same chain transitive component of φ, then

$$L(p, x(p)) = L(p, y(p)).$$

And if $x(p)$ and $y(p)$ belong to the different chain transitive components, then

$$L(p, x(p)) \neq L(p, y(p)).$$

3. Chain recurrent sets for nonautonomous dynamical systems

First we present some Lemmas related to the NDS.

Lemma 3.1. *Assume that $U(p)$ is a given pre-attractor and $\bigcup_{t\geq\tau(p)} \varphi(t,\theta_{-t}p)U(\theta_{-t}p)$ is pre-compact. Then $A(p)$ is a pullback attractor with respect to $\mathcal{D} = \{D(p) \mid D(p)$ is a closed nonempty subset of $U(p)$ for every $p \in P\}$.*

Proof. Denote $U(\tau(p)) = \overline{\bigcup_{t\geq\tau(p)} \varphi(t,\theta_{-t}p)U(\theta_{-t}p)}$. It follows that

$$A(p) = \bigcap_{n\in\mathbb{N}} U(n\tau(p)). \tag{3.1}$$

Therefore $A(p)$ is a nonempty compact set. The invariance of $A(p)$ can be shown as in the proof of invariance for the Ω-limit set of a random set [12].

Moreover, for any $D(p) \in \mathcal{D}$,

$$\lim_{t\to\infty} dist_X(\varphi(t,\theta_{-t}p)D(\theta_{-t}p), A(p)) = 0,$$

which shows that $A(p)$ is the pullback attractor with respect to $\mathcal{D}$. □

Remark 3.2. *In the definition of* (2.3)*, the local attractor needs not be compact and it can be allowed as an empty set. It is known that the attractor is also noncompact in* [1]*. But when the state space is compact, the local attractor is the pullback attractor. For more details about pullback attractor see* [2, 4, 29].

In the random case, the pre-attractor can be selected as a forward invariant open set and repeller is a forward invariant closed set [20, 21]. The following Lemma 3.3, which can be proved as in [20, 21], shows that the basin of attraction is a backward invariant open set and the repeller is forward invariant closed set.

Lemma 3.3. *The basin of attraction $B(A,U)$ is a backward invariant open set and the repeller R is a forward invariant closed set.*

Lemma 3.4. *If $x(p) \in B(A,U)(p)$ and $x(p)$ is the chain recurrent variable for $p \in P$, then $x(p) \in A(p)$ for $p \in P$, where $B(A,U)(p)$ is the basin of attraction determined by $U(p)$ and $A(p)$.*

Proof. The idea is from [6, 16]. For $\tau(p)$ satisfies (2.2), we have

$$\bigcup_{t\geq\tau(p)}\varphi(t,\theta_{-t}p)U(\theta_{-t}p)\subset U(p).$$

We need to show that there exists an $\varepsilon(p,x)>0$ such that $\varepsilon(p,x)\leq 1$ and

$$B(\varphi(t,\theta_{-t}p)x(\theta_{-t}p),\varepsilon(p,\varphi(t,\theta_{-t}p)x(\theta_{-t}p)))\subset U(p) \tag{3.2}$$

for all $x(p)\in U(p)$ and $t\geq\tau(p)$.

Let us construct such a function ε. Define $\delta(p,x)$ by

$$\delta(p,x)=\frac{1}{2}\{d_X(x,\bigcup_{t\geq\tau(p)}\varphi(t,\theta_{-t}p)U(\theta_{-t}p)+d_X(x,X-U(p))\}.$$

Then $\delta(p,x)>0$ since $x\notin X-U(p)$ if $x\in\bigcup_{t\geq\tau(p)}\varphi(t,\theta_{-t}p)U(\theta_{-t}p)\subset U(p)$. Let $x(p)\in U(p)$ and $t\geq\tau(p)$. For any $y(p)\in B(\varphi(t,\theta_{-t}p)x(\theta_{-t}p),\delta(p,\varphi(t,\theta_{-t}p)x(\theta_{-t}p)))$,

$$d_X(\varphi(t,\theta_{-t}p)x(\theta_{-t}p),y(p))<\delta(p,\varphi(t,\theta_{-t}p)x(\theta_{-t}p))=\frac{1}{2}d_X(\varphi(t,\theta_{-t}p)x(\theta_{-t}p),X-U(p)).$$

Thus we have

$$\begin{aligned}&2d_X(\varphi(t,\theta_{-t}p)x(\theta_{-t}p),y(p))<d_X(\varphi(t,\theta_{-t}p)x(\theta_{-t}p),X-U(p))\\&\leq d_X(\varphi(t,\theta_{-t}p)x(\theta_{-t}p),y(p))+d_X(y(p),X-U(p)).\end{aligned}$$

Since $d_X(y(p),X-U(p))>d_X(\varphi(t,\theta_{-t}p)x(\theta_{-t}p),y(p))\geq 0$, we have $y(p)\in U(p)$. Hence

$$B(\varphi(t,\theta_{-t}p)x(\theta_{-t}p),\varepsilon(p,\varphi(t,\theta_{-t}p)x(\theta_{-t}p)))\subset U(p).$$

and $\varepsilon=\min\{\delta,1\}$ is the desired function.

Let m, n be positive integers. Select $\varepsilon>0$ satisfies (3.2) for $x(p)\in U(p)$, $p\in P$. If $x(p)\in CR_\varphi(p)$, there is an $(\frac{\varepsilon}{n},m\tau)(p)$-chain $\{x_1(p),\cdots,x_k(p),\ x_{k+1}(p);t_1,\cdots,t_k\}$ from $x(p)$ back to $x(p)$. Since

$$\begin{aligned}&d(\varphi(t_1,\theta_{-t_1}p)\circ x_1(\theta_{-t_1}p),x_2(p))\\<\ &\frac{1}{n}\varepsilon(p,\varphi(t_1,\theta_{-t_1}p)x_1(\theta_{-t_1}p))\\\leq\ &\varepsilon(p,\varphi(t_1,\theta_{-t_1}p)x_1(\theta_{-t_1}p)).\end{aligned}$$

we have

$$x_2(p)\in B(\varphi(t_1,\theta_{-t_1}p)x(\theta_{-t_1}p),\varepsilon(p,\varphi(t_1,\theta_{-t_1}p)x(\theta_{-t_1}p)))\subset U(p)$$

by (3.2). Thus $x_k(p)\in U(p)$ by induction. Since

$$d(\varphi(t_k,\theta_{-t_k}p)x_k(\theta_{-t_k}p),x_{k+1}(p))<\frac{1}{n}\varepsilon(p,\varphi(t_k,\theta_{-t_k}p)x_k(\theta_{-t_k}p))\leq\frac{1}{n},$$

we obtain

$$d(x(p), \bigcup_{t\geq m\tau(p)} \varphi(t,\theta_{-t}p)U(\theta_{-t}p)) \leq d(x(p), \varphi(t_k,\theta_{-t_k}p)x_k(\theta_{-t_k}p)) < \frac{1}{n}.$$

Thus

$$d(x(p), \bigcup_{t\geq m\tau(p)} \varphi(t,\theta_{-t}p)U(\theta_{-t}p)) = 0.$$

It follows that $x(p) \in Cl \bigcup_{t\geq m\tau(p)} \varphi(t,\theta_{-t}p)U(\theta_{-t}p)$. This implies

$$x(p) \in \bigcap_{m\in\mathbb{N}} Cl \bigcup_{t\geq m\tau(p)} \varphi(t,\theta_{-t}p)U(\theta_{-t}(p)) = A(p).$$

Now $x(p) \in B(A,U)(p)$, $p \in P$, which implies there exists $s \geq 0$ such that

$$\varphi(s,p)x(p) \in U(\theta_s p) \tag{3.3}$$

for fixed $p \in P$. By a same method that given in [7, 16], we can conclude that CR_φ is forward invariant. Hence $\varphi(s,p)x(p) \in CR_\varphi(\theta_s p)$. Combining the above prove process we have

$$\varphi(s,p)x(p) \in A(\theta_s p). \tag{3.4}$$

Let $Y = \{t \geq 0 \mid \pi(t,(p,x(p))) \in U\}$ and $Z = \{t \geq 0 \mid \pi(t,(p,x(p))) \in A\}$. From (3.3) and (3.4) we know that $Y \neq \emptyset$ and $Z \neq \emptyset$. By the continuity of π, Y is open in $[0,+\infty)$, while Z is closed. So $Y = Z = [0,+\infty)$, which shows $x(p) \in A(p)$. □

We now present the following result on chain recurrent set.

Theorem 3.5. *(Chain recurrent set for NDS)*
Let $U(p)$ be an arbitrary pre-attractor, $A(p)$ be the local attractor determined by $U(p)$, and $B(A,U)(p)$ be the basin of attraction determined by $U(P)$ and $A(p)$. Then the following decomposition holds:

$$X - CR_\varphi(p) = \bigcup [B(A,U)(p) - A(p)],$$

where the union is taken over all local attractors $A(p)$ determined by pre-attractors.

Proof. Suppose x is a map from P to X, the function $\varepsilon(p,x)>0$ and $\tau(p)>0$. Define

$$\begin{aligned}
U_1(p): &= \bigcup_{t\geq\tau(p)} B(\varphi(t,\theta_{-t}p)x(\theta_{-t}p),\varepsilon(p,\varphi(t,\theta_{-t}p)x(\theta_{-t}p))),\\
U_2(p): &= \bigcup_{t\geq\tau(p)}\bigcup_{y(p)\in U_1(p)} B(\varphi(t,\theta_{-t}p)y(\theta_{-t}p),\varepsilon(p,\varphi(t,\theta_{-t}p)y(\theta_{-t}p))),\\
&\cdots\\
U_n(p): &= \bigcup_{t\geq\tau(p)}\bigcup_{y(p)\in U_{n-1}(p)} B(\varphi(t,\theta_{-t}p)y(\theta_{-t}p),\varepsilon(p,\varphi(t,\theta_{-t}p)y(\theta_{-t}p))),\\
&\cdots.
\end{aligned}$$

Since $U_n(p)$ $(n\geq 1)$ are all open sets. So the set

$$U_x(p) := \bigcup_{n\in\mathbb{N}} U_n(p) \tag{3.5}$$

is an open set. From the construction of $U_x(p)$ we see that $U_x(p)$ is the set of all possible end points of $(\varepsilon,\tau)(p)$-chains that begin at $x(p)$. In the following we prove that $U_x(p)$ is a pre-attractor and it determines a local attractor $A_x(p)$. Since there exists a map $\delta: P\times X\to(0,+\infty)$ with $\delta\leq\frac{\varepsilon}{2}$ such that $\varepsilon(p,y(p))>\frac{1}{2}\varepsilon(p,x(p))$ when $d(x(p),y(p))<\delta(p,x(p))$. For $B(y(p),\delta(p,y(p)))\bigcap\varphi(t,\theta_{-t}p)U_x(\theta_{-t}p)\neq\emptyset$, $t\geq\tau(p)$. There exists $z(p)\in U_x(p)$ such that

$$d(\varphi(t,\theta_{-t}p)z(\theta_{-t}p),y(p))<\delta(p,y(p)),\quad t\geq\tau(p).$$

Since $d(\varphi(t,\theta_{-t}p)z(\theta_{-t}p),y(p))<\delta(p,y(p))$, we have $\varepsilon(p,\varphi(t,\theta_{-t}p)z(\theta_{-t}p))>\frac{1}{2}\varepsilon(p,y(p))$. Thus

$$d(\varphi(t,\theta_{-t}p)z(\theta_{-t}p),y(p))<\delta(p,y(p))\leq\frac{1}{2}\varepsilon(p,y(p))<\varepsilon(p,\varphi(t,\theta_{-t}p)z(\theta_{-t}p)).$$

Hence there exists an $(\varepsilon,\tau)(p)$-chain from $x(p)$ to $y(p)$. This means that

$$\overline{\bigcup_{t\geq\tau(p)}\varphi(t,\theta_{-t}p)U_x(\theta_{-t}p)}\subset U_x(p).$$

If $x(p)\in X-CR_\varphi(p)$, then for arbitrary $\varepsilon(p,x)>0$ and $\tau(p)>0$ there exists no $(\varepsilon,\tau)(p)$-chain begins and ends at $x(p)$. Take U_x defined by (3.5), then by the construction of U_x, it is easy to see that $x(p)\in B(A_x,U_x)(p)$ and $x(p)\notin U_x(p)$. Hence

$$x(p)\in B(A_x,U_x)(p)-A_x(p)$$

for $p\in P$. So

$$X-CR_\varphi(p)\subset\bigcup[B(A,U)(p)-A(p)]. \tag{3.6}$$

If $x(p)$ is a chain recurrent variable and $x(p)\in B(A,U)(p)$, then by Lemma 3.4, we have $x(p)\in A(p)$. Hence

$$x(p)\in X-CR_\varphi(p)\quad\text{whenever}\quad x(p)\in B(A,U)(p)-A(p).$$

So

$$\bigcup[B(A,U)(p) - A(p)] \subset X - CR_\varphi(p). \tag{3.7}$$

Therefore, by (3.6) and (3.7), we obtain that

$$X - CR_\varphi(p) = \bigcup[B(A,U)(p) - A(p)].$$

□

Corollary 3.6. *Assume that $U(p)$ is a pre-attractor, $A(p)$ is the local attractor determined by $U(p)$, and $R(p)$ is the repeller corresponding to $A(p)$ with respect to $U(p)$. Then*

$$CR_\varphi(p) = \bigcap[A(p) \bigcup R(p)],$$

where the intersection is taken over all local attractrs.

Lemma 3.7. *Let U and A be nonautonomous sets. If U is the pre-attractor for the skew-product system, then*

$$A(p) = \bigcap_{t \geq T} \overline{\bigcup_{s \geq t} \varphi(s, \theta_{-s}p, U(\theta_{-s}p))}$$

is the local attractor with the property $\bigcup\limits_{p \in P} \{p\} \times A(p) \subset A$, where we denote $(p, \emptyset) = \emptyset$.

Proof. Since U is the pre-attractor for the skew-product system, there exists $T \geq 0$ such that $\overline{\bigcup\limits_{s \geq T} \pi(s, U)} \subset U$. Suppose $y \in \overline{\bigcup\limits_{s \geq T} \varphi(s, \theta_{-s}p, U(\theta_{-s}p))}$, there exists $s_n \geq T$ and $x_n \in U(\theta_{-s_n}p)$ such that

$$\begin{aligned}(p, y) &= \lim_{n \to \infty} (p, \varphi(s_n, \theta_{-s_n}p, x_n)) \\ &= \lim_{n \to \infty} \pi(s_n, (\theta_{-s_n}p, x_n)) \\ &\in U = \bigcup_{p \in P} \{p\} \times U(p). \end{aligned} \tag{3.8}$$

We thus conclude that $y \in U(p)$. Thus $U(p)$ is the pre-attractor and $A(p)$ is the corresponding local attractor.

Assume that $A = \bigcup\limits_{p \in P} \{p\} \times A'(p)$ with $A'(p) = \{x \mid (p, x) \in A \text{ for fixed } p\}$. If $A(p) \neq \emptyset$, let the sequence $s_n \to +\infty$ in (3.8). As in the proof of (3.8), we see that $A(p) \subset A'(p)$. If $A(p) = \emptyset$, we also have $A(p) \subset A'(p)$. Hence $\bigcup\limits_{p \in P} \{p\} \times A(p) \subset A$. □

Corollary 3.8. *With the convention $(p, \emptyset) = \emptyset$, we have the following relation: $\bigcup\limits_{p \in P} \{p\} \times CR_\varphi(p) \subset CR(\pi)$.*

Proof. From Theorem 3.5, Lemma 3.7 and the fact that $B(A,U) = \bigcup_{p\in P} \{p\} \times B(A,U)(p)$, the required relation follows. $\square$

4. Complete Lyapunov functions for nonautonomous dynamical systems

If X is a separable metric space, then we take $\{x_j\}_{j=1}^{\infty}$ as a countable dense subset of X. For $p \in P$, we define $x_j(p) = x_j$, $\varepsilon_j(p,x) = \varepsilon_j$ and $\tau_j(p) = \tau_j$, where $\{\varepsilon_j\}_{j=1}^{\infty} \in \mathbb{Q}^+$ and $\{\tau_j\}_{j=1}^{\infty} \in \mathbb{Q}^+$. By the construction of (3.5), we obtain countable local attractors which we denote as $\{A_n(p)\}_{n=1}^{\infty}$.

Lemma 4.1. *Assume that $(A(p), R(p))$ is a given attractor-repeller pair and $A(p) \in \{A_n(p)\}_{n=1}^{\infty}$. Then there exists a function l for $(A(p), R(p))$ such that $l(p,x)$ is continuous with respect to $x \in X$ and possesses the following properties:*
(i) $l(p,x) = 0$ when $x \in A(p)$, and $l(p,x) = 1$ when $x \in R(p)$, $p \in P$;
(ii) For $\forall x \in X\backslash(A(p) \bigcup R(p))$ and for $\forall t > 0$: $1 > l(p,x) > l(\theta_t p, \varphi(t,p)x) > 0$.

Proof. From (3.5), we know that $U(\theta_s p) = U(p)$, $A(\theta_s p) = A(p)$ and $R(\theta_s p) = R(p)$ for $s \in \mathbb{T}$. Let
$$\lambda(p,x) := \frac{dist_X(x, A(p))}{dist_X(x,A(p)) + dist_X(x,R(p))}.$$
We set $dist_X(x,R(p)) = 1$ if $R(p) = \emptyset$, and $dist_X(x,A(p)) = 1$ if $A(p) = \emptyset$.

Then the function λ is continuous with respect to $x \in X$ and
$$\lambda(p,x) = \begin{cases} 0, & x \in A(p); \\ 1, & x \in R(p); \end{cases}$$

Let $g(p,x) = \sup_{t\geq 0} \lambda(\theta_t p, \varphi(t,p)x)$. In the following we prove that $g(p,x)$ is continuous with respect to $x \in X$. Since $1 \geq g(p,x) \geq \lambda(p,x) = 1$ for $x \in R(p)$, $g(p,x)$ is continuous at $x \in R(p)$. For $x \in X\backslash R(p)$, there exists $t_0 \geq 0$ such that $\varphi(t_0,p,x) \in U(\theta_{t_0}p)$. We also have $\varphi(t_0, \theta_{-t-t_0}p, x) \in U(p)$ for $t \in \mathbb{T}$. So
$$\lim_{t\to\infty} dist_X(\varphi(t,p,x), A(\theta_t p)) \leq \lim_{t\to\infty} dist_X(\overline{\varphi(t,\theta_{-t}p)U(\theta_{-t}p)}, A(p)) = 0. \tag{4.1}$$

This implies that $g(p,x) = \sup_{t_0 \geq t\geq 0} \lambda(\theta_t p, \varphi(t,p)x)$ for some $t_0 > 0$. The continuity of $g(p,x)$ for fixed $p \in P$ follows from the continuity of $g(p,x) = \sup_{t_0\geq t\geq 0} \lambda(\theta_t p, \varphi(t,p)x)$ for fixed $p \in P$. By definition we have $g(\theta_t p, \varphi(t,p)x) \leq g(p,x)$.

The function l is defined by $l(p,x) = \int_0^{+\infty} e^{-t} g(\theta_t p, \varphi(t,p)x)dt$. Since $A(p)$ and $R(p)$ are forward invariant, we conclude that $l(p,x) = 0$ for $x \in A(p)$, and $l(p,x) = 1$ for

$x \in R(p)$. If $x \in X \backslash (A(p) \bigcup R(p))$, we define $l(\theta_{t_1} p, \varphi(t_1, p)x) = l(p, x)$ for some $t_1 > 0$. Hence $g(\theta_t p, \varphi(t, p)x) = g(\theta_{t+t_1} p, \varphi(t + t_1, p)x)$ for all $t \geq 0$, from which we see that

$$g(x, p) = g(\theta_{nt_1} p, \varphi(nt_1, p)x)$$

for all $n \in \mathbb{N}$. Let $n \to \infty$, we arrive at a contradiction to (4.1). Hence we have

$$l(\theta_t p, \varphi(t, \theta_{-t} p)x) < l(p, x).$$

Obviously, $l(p, x)$ is continuous with respect to $x \in X$. □

Remark 4.2. *In this proof, we have used a similar construction as in* [22, 25, 27]*, which was originated from* [8]*.*

Assume that $l_n(p, x)$ is the Lyapunov function determined by the attractor-repeller pair $(A_n(p),\ R_n(p))$. Define

$$L(p, x) = \sum_{n=1}^{\infty} \frac{2l_n(p, x)}{3^n}. \tag{4.2}$$

We now show that $L(p, x)$ defined by (4.2) is a complete Lyapunov function for φ.

Theorem 4.3. *(Complete Lyapunov function for NDS)*
The function defined by (4.2) *is a complete Lyapunov function for the NDS* φ*.*

Proof. We show that all conditions in Definition 2.7 are satisfied.

(a) If $x(p) \in CR_\varphi(p)$, then we have

$$l_n(p, x(p)) = l_n(\theta_t p, \varphi(t, p)x(p)), \quad \forall t \geq 1,$$

which takes value 0 or 1 for each $n \in \mathbb{N}$ by Corollary 3.6 and Lemma 4.1.
(b) By Lemma 4.1, we have

$$l_n(\theta_t p, \varphi(t, p)x) < l_n(p, x), \qquad \forall t > 0, n \in \mathbb{N}$$

for $x \in X \backslash (A_n(p) \bigcup R_n(p))$. If $x \in A_n(p) \bigcup R_n(p)$, then $\varphi(t, p)x \in A_n(\theta_t p) \bigcup R_n(\theta_t p)$. So for $x \in X - CR_\varphi(p)$,

$$L(\theta_t p, \varphi(t, p)x) \leq L(p, x), \qquad \forall t > 0, n \in \mathbb{N}.$$

(c) This is due to the fact that $L(p, CR_\varphi(p))$ is a subset of the Cantor middle-third set.
(d) If $x(p) \in A_n(p)$, with $x(p)$ and $y(p)$ belonging to the same chain transitive component, then $y(p) \in U_n(p) \subset B(A_n, U_n)(p)$ for $p \in P$. Therefore $y(p) \in A_n(p)$ for $p \in P$ by Lemma 3.4. Similarly, if $x(p) \in R_n(p)$ then $y(p) \in R_n(p)$. It is clear that $L(p, \cdot)$ is constant on each chain transitive component and $L(p, \cdot)$ takes different values on different transitive components.

This completes the proof. □

Remark 4.4. *The complete Lyapunov function obtained in Lemma 4.3 is weaker than that in the autonomous case. It is nonincreasing along orbits of the skew-product flow* $\pi(t,(p,x))$.

Proof of Theorem 1.1:

Making use of the above Theorems 3.5 and 4.3 for chain recurrent set and complete Lyapunov function, respectively, we obtain the decomposition in Theorem 1.1.

5. Applications

In this section we consider a few examples to illustrate the applications of the decomposition result in Theorem 1.1.

Example 5.1. *Consider the differential equation* [19]

$$\dot{x} = 2tx. \tag{5.1}$$

The solution is $x(t,t_0,x_0) = x_0 e^{t^2 - t_0^2}$, *where* $t \geq t_0$, *and the cocycle mapping is*

$$\varphi(t,t_0,x_0) = x_0 e^{(t+t_0)^2 - t_0^2}, \qquad t \geq 0.$$

Here the driving space is $P = \mathbb{R}$ *with element* $p = t_0$, *the shift map is* $\theta_t t_0 = t + t_0$ *and the state space is* $X = \mathbb{R}$. *The pre-attractor is selected as* $U = (-1,1)$. *The corresponding local attractor is* $A(p) = \{0\}$. *From the definition of* φ *we see that the basin of attraction is* $B(A,U)(p) = \mathbb{R}$. *So the chain recurrent set is* $\{0\}$. *Moreover, the complete Lyapunov function is nonincreasing outside* $X - \{0\}$.

We revise the example in [11] to fit our purpose here.

Example 5.2. *Consider the base space* $P = S^1$ *and the state space* $X = S^1$. *Define a shift map* $\theta_t p = p + t$. *Introduce a homeomorphism* $\psi(p) : S^1 \to S^1$ *by* $\psi(p)x = x + p$. *We consider a NDS defined by*

$$\varphi(t,p) = \psi(\theta_t p) \circ \varphi_0(t) \circ \psi^{-1}(p),$$

where φ_0 *is the semiflow on* S^1 *determined by the equation*

$$\dot{x} = -\cos x.$$

Then the NDS φ *has no nontrivial local attractor. Hence by the decomposition result in Theorem 1.1, the chain recurrent set is* X. *This means that for any* $x(p) \in X$, *there exists an* $(\varepsilon, T)(p)$*-chain, beginning and ending at* $x(p)$ *for* $p \in P$.

The following Lorenz system under time-dependent forcing was considered in [3].

Example 5.3. *Let Ω be a compact metric space, $\mathbb{R} = (-\infty, +\infty)$, $(\Omega, \mathbb{R}, \sigma)$ be a dynamical system on Ω, and H be a Hilbert space. We denote $L(H)$ and $L^2(H)$ as the spaces of all linear, bilinear, respectively, endomorphisms on H. Let $C(\Omega, H)$ be the space of all continuous functions $f : \Omega \to H$, endowed with the topology of uniform convergence. Consider the non-autonomous Lorenz system*

$$u' = A(\omega t)u + B(\omega t)(u, u) + f(\omega t), \quad \omega \in \Omega, \tag{5.2}$$

where $\omega t := \sigma(t, \omega)$, $A \in C(\Omega, L(H))$, $B \in C(\Omega, H)$, and $f \in C(\Omega, H)$. Moreover, we assume (5.2) *satisfies the following conditions:*
*(*a) there exists $\alpha > 0$ such that

$$Re\langle A(\omega)u, u\rangle \leq -\alpha|u|^2$$

for all $\omega \in \Omega$ and $u \in H$, where $|\cdot|$ is a norm in H, generated by the scalar product $\langle\cdot, \cdot\rangle$;
(b)

$$Re\langle B(\omega)(u, v), w\rangle = -Re\langle B(\omega)(u, w), v\rangle$$

for every $u, v, w \in H$ and $\omega \in \Omega$. It can be shown that there exists a global solution $\varphi(t, x, \omega)$ of (5.2) on $\mathbb{R}^+$, which defines a nonautonomous dynamical system φ.

Recall that the dynamical system $(\Omega, \mathbb{R}, \theta)$ is called asymptotically compact if for any positively invariant bounded set $A \subset X$, there is a compact $K_A \subset X$ such that

$$\lim_{t\to+\infty} dist_\Omega(\theta(t, A), K_A) = 0.$$

The NDS φ is called asymptotic compact if the associated skew-product flow, on $P \times X$, is asymptotic compact.

If $(\|f\|C_B)/\alpha^2 < 1$, where $C_B := \sup\{|B(\omega)(u, v)| : \omega \in \Omega, u, v \in H, |u| \leq 1, |v| \leq 1\}$ and the system (5.2) is asymptotic compact, then there exists a map $x : \Omega \to H$ such that $x(\omega t) = \varphi(t, x(\omega), \omega)$ for all $\omega \in \Omega$ and $t \in \mathbb{R}^+$. Here we have $P = \Omega$ and $X = H$. So the map x is chain recurrent with respect to Ω. Therefore using the decomposition result in Theorem 1.1, we can decompose the space H into two nontrivial parts. One is the chain recurrent part and the other is the gradient-like part.

Remark 5.4. *Note that a stationary orbit (or stationary solution) is chain recurrent. If we know that there exists a stationary solution in a nonautonomous differential equation, we can conclude that the chain recurrent set is not empty. This is one way to demonstrate that the chain recurrent set is not empty.*

The following example is the nonautonomous Navier-Stokes equation, which was also considered in [4, 26].

Example 5.5. *Consider the 2-dimensional Navier-Stokes equation*

$$u_t + \sum_{i=1}^{2} u_i \partial_i u = \nu \triangle u + \nabla p + F(t) \tag{5.3}$$
$$div u = 0, \quad u|_{\partial D} = 0,$$

where $u = (u_1, u_2)$ *is the velocity field,* $F(x, y, t) = (F_1, F_2)$ *is the external forcing, and* D *is an open bounded fluid domain with smooth boundary* $\partial D \in C^2$*. We denote by* H *and* V *the closures of the linear space* $\{u \mid u \in C_0^\infty(D)^2\}$ *in* $L^2(D)^2$ *and* $H_0^1(D)^2$*, respectively. We also denote by* Pr *the corresponding orthogonal projection* $Pr : L^2(D)^2 \to H$*. We further set*

$$A := -\nu Pr \triangle, \quad B(u, u) := Pr(\sum_{i=1}^{2} u_i \partial v)$$

Applying the orthogonal projection Pr*, we rewrite the Navier Stokes equation* (5.3) *in the operator form*

$$\frac{du}{dt} + Au + B(u, u) = f(t), \quad u(0) = u_0 \in H, \tag{5.4}$$

Here $X = H$*, which is a Hilbert space. Now suppose* f *is a periodic function in* $C(\mathbb{R}, H)$ *and define* $\theta_t f(\cdot) := f(\cdot + t)$*. Then* $P = \bigcup_{t \in \mathbb{R}} \theta_t f$ *is a compact subset of* $C(\mathbb{R}, H)$*. Then* $\varphi(t, u, p) := u(t, u, p)$ *is continuous from* $\mathbb{R}^+ \times H \times C(\mathbb{R}, H) \to H$*. Then the nonautonomous dynamical system* $(H, \varphi, (P, \mathbb{R}, \theta))$ *generated by the Navier-Stokes equation* (5.4) *with periodic forcing term in* $C(\mathbb{R}, H)$ *has a pullback attractor. It is the nontrivial local attractor. By Theorem 3.5, we conclude that the chain recurrent set is not empty. We can also decomposition the space* H *into two parts: the chain recurrent part* $CR_\varphi(p)$ *and gradient-like part* $H - CR_\varphi(p)$*.*

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(X. Chen) School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan 430074, China

E-mail address, X. Chen: `chenxiao002214336@yahoo.cn`

(J. Duan) Department of Applied Mathematics, Illinois Institute of Technology, Chicago, IL 60616, USA

E-mail address, J. Duan: `duan@iit.edu`